

Lecture 51

Line integral: Any integral which is evaluated along a curve is called a line integral.

Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be the given equation of the curve C , joining the two points A at $t = t_1$ and B at $t = t_2$. Suppose $\vec{F}(x, y, z) = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ is a vector function and continuous along C . If s denotes the arc length of C , then $\frac{d\vec{r}}{ds} = \hat{t}$, unit tangent vector to C at the point P . Then the component of $\vec{F}$ along the tangent is $\vec{F} \cdot \frac{d\vec{r}}{ds}$. The integral of $\vec{F} \cdot \frac{d\vec{r}}{ds}$ along C from A to B is given by

$$\int_A^B \left[\vec{F} \cdot \frac{d\vec{r}}{ds} \right] ds = \int_A^B \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_{t_1}^{t_2} \left[F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} \right] dt.$$

Circulation: If C is a simple closed curve (i.e., a curve which does not intersect itself anywhere), then the tangent line integral of $\vec{F}$ around C is called the circulation of $\vec{F}$ about C . It is often denoted by

$$\Gamma = \oint_C \vec{F} \cdot d\vec{r} = \oint_C (F_x dx + F_y dy + F_z dz).$$

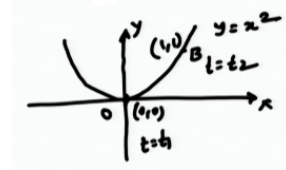
Example. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = x^2\hat{i} + y^3\hat{j}$ and C is the arc of the curve $y = x^2$ in xy -plane from $(0, 0)$ to $(1, 1)$.

Solution: C is the curve $y = x^2$ joining $(0, 0)$ to $(1, 1)$. The parametric form of the curve is given by

$$x = t, \quad y = t^2, \quad \text{where } 0 \leq t \leq 1.$$

The line integral

$$\begin{aligned} I &= \int_C \vec{F} \cdot d\vec{r} = \int_C (x^2\hat{i} + y^3\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) \\ &= \int_0^1 (t^2\hat{i} + t^6\hat{j}) \cdot (\hat{i} + 2t\hat{j}) dt \\ &= \int_0^1 (t^2 + 2t^7) dt \\ &= \left[\frac{t^3}{3} + \frac{2t^8}{8} \right]_0^1 = \frac{7}{12}. \end{aligned}$$



Example. If $\vec{F}(x, y) = 3xy\hat{i} - y^2\hat{j}$, then evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve in the xy -plane $x^2 + y^2 = 25$ from $(5, 0)$ to $(0, 5)$.

Solution: C is the line segment of the circle $x^2 + y^2 = 25$ joining $(5, 0)$ to $(0, 5)$. The parametric form of the curve is given by

$$x = 5 \cos t, \quad y = 5 \sin t, \quad \text{where } 0 \leq t \leq \frac{\pi}{2}.$$

The line integral

$$\begin{aligned} I &= \int_C \vec{F} \cdot d\vec{r} = \int_C (3xy\hat{i} - y^2\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) \\ &= \int_0^{\frac{\pi}{2}} (3 \times 25 \cos t \sin t \hat{i} - 25 \sin^2 t \hat{j}) \cdot (-\sin t \hat{i} + \cos t \hat{j}) dt \\ &= \int_0^{\frac{\pi}{2}} (-75 \sin^2 t \cos t - 25 \sin^2 t \cos t) dt \\ &= \int_0^{\frac{\pi}{2}} -100 \sin^2 t d(\sin t) \\ &= \left[-\frac{100 \sin^3 t}{3} \right]_0^{\frac{\pi}{2}} = -\frac{100}{3}. \end{aligned}$$

Example. Evaluate $\int_C (x dy - y dx)$ around the circle $x^2 + y^2 = 1$.

Solution: C is the circle $x^2 + y^2 = 1$. The parametric form of the curve is given by

$$x = \cos t, \quad y = \sin t, \quad \text{where } 0 \leq t \leq 2\pi.$$

Therefore, the line integral

$$\begin{aligned} I &= \int_C (x dy - y dx) \\ &= \int_0^{2\pi} (\cos t \times \cos t + \sin t \times \sin t) dt \\ &= \int_0^{2\pi} dt = 2\pi. \end{aligned}$$

Example. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = xy\hat{i} + yz\hat{j} + zx\hat{k}$ and the given curve $\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$, where t runs from -1 to 1 .

Solution: C is the curve $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ where the parametric form of the curve is given by

$$x = t, \quad y = t^2, \quad z = t^3, \quad \text{where } -1 \leq t \leq 1.$$

Therefore, the line integral

$$\begin{aligned} I &= \int_C \vec{F} \cdot d\vec{r} = \int_C (xy\hat{i} + yz\hat{j} + zx\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ &= \int_{-1}^1 (t^3\hat{i} + t^5\hat{j} + t^4\hat{k}) \cdot (\hat{i} + 2t\hat{j} + 3t^2\hat{k}) dt \\ &= \int_{-1}^1 (t^3 + 5t^6) dt \\ &= \left[\frac{t^4}{4} + \frac{5t^7}{7} \right]_{-1}^1 = \frac{10}{7}. \end{aligned}$$

Example. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = (x^2 + y^2)\hat{i} - 2xy\hat{j}$, where C is the rectangle in the xy -plane $y = 0$, $x = 0$, $y = b$, and $x = a$.

Solution:

$$\int_C \vec{F} \cdot d\vec{r} = \underbrace{\int_{C_1} \vec{F} \cdot d\vec{r}}_{I_1} + \underbrace{\int_{C_2} \vec{F} \cdot d\vec{r}}_{I_2} + \underbrace{\int_{C_3} \vec{F} \cdot d\vec{r}}_{I_3} + \underbrace{\int_{C_4} \vec{F} \cdot d\vec{r}}_{I_4}$$

Along the curve C_1 : $y = 0$, and x varies from 0 to a .

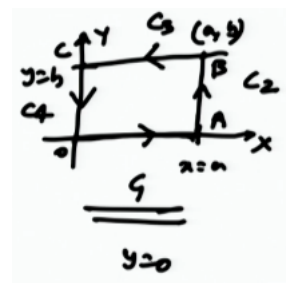
Therefore,

$$\begin{aligned} I_1 &= \int_0^a (x^2\hat{i} + 0\hat{j}) \cdot (dx\hat{i} + 0\hat{j}) \\ &= \int_0^a x^2 dx = \frac{a^3}{3}. \end{aligned}$$

Along the curve C_2 : $x = a$, and y varies from 0 to b .

Therefore,

$$\begin{aligned} I_2 &= \int_0^b ((a^2 + y^2)\hat{i} - 2ay\hat{j}) \cdot (0\hat{i} + dy\hat{j}) \\ &= \int_0^b -2ay dy = -ab^2. \end{aligned}$$



Along the curve C_3 : $y = b$, and x varies from a to 0. Therefore,

$$\begin{aligned} I_3 &= \int_a^0 ((x^2 + b^2) \hat{i} - 2bx \hat{j}) \cdot (dx \hat{i} + 0 \hat{j}) \\ &= \int_a^0 (x^2 + b^2) dx = -\frac{a^3}{3} - ab^2. \end{aligned}$$

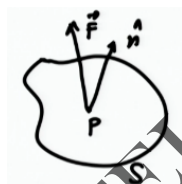
Along the curve C_4 : $x = 0$, and y varies from b to 0. Therefore,

$$\begin{aligned} I_3 &= \int_b^0 (y^2 \hat{i} + 0 \hat{j}) \cdot (0 \hat{i} + dy \hat{j}) \\ &= 0. \end{aligned}$$

Therefore, $\int_C \vec{F} \cdot d\vec{r} = -2ab^2$.

Surface integral: Any integral which is evaluated over a surface S is called the surface integral.

$$I_S = \iint_S \vec{F} \cdot \hat{n} ds = \iint_S \vec{F} \cdot d\vec{s}.$$



To evaluate the integral, first we take the projection,

$$\begin{aligned} I_S &= \iint_S \vec{F} \cdot \hat{n} ds = \iint_S \vec{F} \cdot \hat{n} \frac{dx dy}{|\hat{n} \cdot \hat{k}|} && \text{on } xy\text{-plane} \\ &= \iint_S \vec{F} \cdot \hat{n} \frac{dy dz}{|\hat{n} \cdot \hat{i}|} && \text{on } yz\text{-plane} \\ &= \iint_S \vec{F} \cdot \hat{n} \frac{dx dz}{|\hat{n} \cdot \hat{j}|} && \text{on } xz\text{-plane} \end{aligned}$$

Lecture 52

Example. Evaluate $\iint_S \vec{F} \cdot \hat{n} ds$, where $\vec{F}(x, y, z) = yz \hat{i} + zx \hat{j} + xy \hat{k}$, and S is the part of surface of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant.

Solution: A vector normal to the surface $S : x^2 + y^2 + z^2 = 1$ is given by $\vec{\nabla} \phi$, where $\phi(x, y, z) = x^2 + y^2 + z^2 - 1$. The normal vector to the surface $\vec{n} = \vec{\nabla} \phi = 2(x \hat{i} + y \hat{j} + z \hat{k})$.

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2(x \hat{i} + y \hat{j} + z \hat{k})}{2\sqrt{x^2 + y^2 + z^2}} = x \hat{i} + y \hat{j} + z \hat{k}.$$

Taking the projection of S into the xy -plane, then

$$\begin{aligned}
 \iint_S \vec{F} \cdot \hat{n} \, ds &= \iint_R (yz \hat{i} + zx \hat{j} + xy \hat{k}) \cdot (x \hat{i} + y \hat{j} + z \hat{k}) \frac{dx \, dy}{|\hat{n} \cdot \hat{k}|} \\
 &= \iint_R 3xyz \frac{dx \, dy}{z} = \iint_R 3xyz \, dx \, dy \\
 &= \int_{r=0}^1 \int_{\theta=0}^{\frac{\pi}{2}} r \cos \theta \, r \sin \theta \, r \, dr \, d\theta \\
 &= \frac{3}{2} \int_{r=0}^1 r^3 \, dr \int_{\theta=0}^{\frac{\pi}{2}} \sin 2\theta \, d\theta \\
 &= \frac{3}{2} \left[\frac{r^4}{4} \right]_0^1 \left[-\frac{\cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} = \frac{3}{8}.
 \end{aligned}$$

Example. Evaluate $\iint_S \vec{F} \cdot \hat{n} \, ds$, where $\vec{F}(x, y, z) = z \hat{i} + x \hat{j} - 3y^2 z \hat{k}$, and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant $z = 0$ and $z = 5$.

Solution: Let us take $\phi(x, y, z) = x^2 + y^2 - 16$, then $\vec{\nabla} \phi = 2(x \hat{i} + y \hat{j})$ and $\hat{n} = \frac{2(x \hat{i} + y \hat{j})}{2\sqrt{x^2 + y^2}} = \frac{x \hat{i} + y \hat{j}}{4}$.

We take R as the projection of S in xz -plane. Then

$$\begin{aligned}
 \iint_S \vec{F} \cdot \hat{n} \, ds &= \iint_R (z \hat{i} + x \hat{j} - 3y^2 z \hat{k}) \cdot \frac{x \hat{i} + y \hat{j}}{4} \frac{dx \, dy}{|\hat{n} \cdot \hat{j}|} \\
 &= \iint_R \frac{(xz + xy) 4 \, dx \, dy}{4y} \quad \text{where } y = \sqrt{16 - x^2} \\
 &= \int_{z=0}^5 \int_{x=0}^4 \frac{xz + x\sqrt{16 - x^2}}{\sqrt{16 - x^2}} \, dx \, dz \\
 &= \int_{z=0}^5 \left[-z\sqrt{16 - x^2} + \frac{x^2}{2} \right]_0^4 \, dz \\
 &= \int_{z=0}^5 (4z + 8) \, dz = 90.
 \end{aligned}$$

Example. Evaluate $\iint_S \vec{F} \cdot \hat{n} \, ds$, where $\vec{F}(x, y, z) = y \hat{i} + 2x \hat{j} - z \hat{k}$, and S is the surface of the plane $x + y = 6$ in the first octant cut by the plane $z = 4$.

Solution: Let us take $\phi(x, y, z) = 2x + y - 6$, then $\vec{n} = \vec{\nabla} \phi = (2\hat{i} + \hat{j})$ and $\hat{n} = \frac{1}{\sqrt{5}}(2\hat{i} + \hat{j})$. Taking the projection on xz -plane,

$$\begin{aligned}
 I_S &= \iint_S \vec{F} \cdot \hat{n} \, ds = \iint_R \frac{1}{\sqrt{5}}(2y + 2x) \frac{dx \, dz}{|\hat{n} \cdot \hat{j}|} \\
 &= \int_{z=0}^4 \int_{x=0}^3 [2x + 2(6 - 2x)] \, dx \, dz \\
 &= 2 \int_{z=0}^4 \int_{x=0}^3 (6 - x) \, dx \, dz \\
 &= 2 \cdot 4 \cdot \left[18 - \frac{9}{2} \right] = 108.
 \end{aligned}$$

Lecture 53

Volume integrals: Suppose V is a volume bounded by surface S . Suppose $f(x, y, z)$ is a scalar function defined on V . Then the volume integral of $f(x, y, z)$ is denoted by

$$I_V = \iiint_V f(x, y, z) dv,$$

where $dv = dx dy dz$. If f is a vector function, say $\vec{f}$, then $\iiint_V \vec{f} \cdot d\vec{v}$ is an example of volume integral.

Example. Evaluate $\iiint_V f(x, y, z) dv$, where $f(x, y, z) = 45x^2y$ and V is the volume of the closed region bounded by $4x + 2y + z = 8$, $x = 0$, $y = 0$, and $z = 0$.

Solution: The region V is the following:

$$z\text{-varies in the range: } 0 \leq z \leq 8 - 4x - 2y,$$

$$y\text{-varies in the range: } 0 \leq y \leq 4 - 2x,$$

$$x\text{-varies in the range: } 0 \leq x \leq 2.$$

$$\begin{aligned} I_V &= \iiint_V f(x, y, z) dv = \int_{x=0}^2 \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} 45x^2y dz dy dx \\ &= 45 \int_{x=0}^2 \int_{y=0}^{4-2x} x^2y(8-4x-2y) dy dx \\ &= 45 \int_{x=0}^2 \frac{x^2}{3}(4-2x)^3 dx = 128. \end{aligned}$$

Example. Evaluate the volume integral $\iiint_V f(x, y, z) dv$, where $f(x, y, z) = xyz$ and V is the volume of the region bounded by the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

Solution: The region V is the following:

$$z\text{-varies in the range: } 0 \leq z \leq \sqrt{1-x^2-y^2},$$

$$y\text{-varies in the range: } 0 \leq y \leq \sqrt{1-x^2},$$

$$x\text{-varies in the range: } 0 \leq x \leq 1.$$

$$\begin{aligned} I_V &= \iiint_V f(x, y, z) dv = \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz dz dy dx \\ &= \frac{1}{2} \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} xy(1-x^2-y^2) dy dx \\ &= \frac{1}{8} \int_{x=0}^1 x(1-x^2)^2 dx = \frac{1}{48}. \end{aligned}$$

Gauss divergence theorem: Suppose V is the volume bounded by a closed piecewise smooth surface S . Let $\vec{f}(x, y, z)$ be a vector function of position which is continuous and has continuous first order partial derivative on V . Then,

$$\iiint_V \operatorname{div} \vec{f} \, ds = \iint_S \vec{f} \cdot \hat{n} \, ds,$$

where $\hat{n}$ is the unit outward drawn normal to S .

Example. If $\vec{f}(x, y, z) = ax\hat{i} + by\hat{j} + cz\hat{k}$, a, b, c are constant, then find the value of $\iint_S \vec{f} \cdot \hat{n} \, ds$, where S is the surface of unit sphere.

Solution: By divergence theorem,

$$\begin{aligned} \iint_S \vec{f} \cdot \hat{n} \, ds &= \iiint_V \operatorname{div} \vec{f} \, dv \\ &= \iiint_V (a + b + c) \, dv = \frac{4\pi}{3} (a + b + c). \end{aligned}$$

Lecture 54

Green's Theorem: Let R be a closed bounded region in xy -plane whose boundary C consists of finitely many smooth curves. Let M and N be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in R . Then,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy = \oint_C (M \, dx + N \, dy),$$

where the line integral being taken along the entire boundary C of R such that R is on the left side as one advances in the direction of integration.



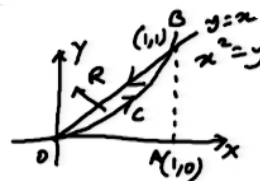
Example. Verify Green's theorem in the plane for $\oint_C ((xy + y^2) \, dx + x^2 \, dy)$, where C is the closed curve of the region bounded by $y = x^2$ and $y = x$.

Solution: Given that $M(x, y) = xy + y^2$ and $N(x, y) = x^2$. To verify Green's theorem, we need to show

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy = \oint_C (M \, dx + N \, dy).$$

$\frac{\partial M}{\partial y} = x + 2y$, $\frac{\partial N}{\partial x} = 2x$. The curve intersect at $(0, 0)$ and $(1, 1)$. Therefore, the left hand side

$$\begin{aligned} \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy &= \int_{x=0}^1 \int_{y=x}^{x^2} (x - 2y) \, dx \, dy \\ &= \int_{x=0}^1 (x^4 - x^3) \, dx = -\frac{1}{20}. \end{aligned}$$



Right hand side implies

$$\begin{aligned}\oint_C (M dx + N dy) &= \oint_C ((xy + y^2) dx + x^2 dy) \\ &= \int_{C_1} ((xy + y^2) dx + x^2 dy) + \int_{C_2} ((xy + y^2) dx + x^2 dy).\end{aligned}$$

Along C_1 : $x^2 = y$ implies $2x dx = dy$. Along C_2 : $x = y$ implies $dx = dy$.

$$\begin{aligned}\oint_C (M dx + N dy) &= \int_0^1 ((x^3 + x^4) dx + x^2 \times 2x dx) + \int_0^1 ((x^2 + x^2) dx + x^2 dx) \\ &= \int_0^1 (x^4 + 3x^3 + 3x^2) dx = -\frac{1}{20}.\end{aligned}$$

Stoke's Theorem: Let S be a piecewise smooth open surface bounded by a piecewise simple closed curve C . Let $\vec{F}(x, y, z)$ be a continuous vector function which has continuous first order partial derivatives in the region of space which contains S in its interior. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n} ds = \iint_S \text{curl } \vec{F} \cdot \vec{n} ds,$$

where $\hat{n}$ is the outward unit normal on S .

Example. Find the value of $\oint_C \vec{r} \cdot d\vec{r}$, where C is any closed curve bounding a surface S .

Solution: $\oint_C \vec{r} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{r}) \cdot \hat{n} ds = \iint_S \vec{0} \cdot \hat{n} ds = 0$.

Example. Verify Stoke's theorem for $\vec{F}(x, y, z) = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution: By Stoke's theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} ds$.

The boundary C of S is the circle in xy -plane of radius unity and center at origin. Let $x = \cos t$, $y = \sin t$, $z = 0$, $t \in [0, 2\pi]$.

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} &= \oint_C [(2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}] \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ &= \int_C (2x - y) dx \\ &= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t) dt = \pi.\end{aligned}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix} = \hat{k}$$

$$\begin{aligned}\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} ds &= \iint_S \hat{k} \cdot \hat{k} ds \\ &= \iint_S ds = \pi.\end{aligned}$$

Lecture 55

Example. Verify Stokes theorem for $\vec{F}(x, y) = (x^2 + y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by $x = \pm a$, $y = 0$, $y = b$.

Solution:

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \iint_S \text{curl } \vec{F} \cdot \vec{n} \, ds \\ &= \int_{x=-a}^a \int_{y=0}^b (-4y\hat{k}) \cdot \hat{k} \, ds = -4ab^2.\end{aligned}$$

Example. By converting the surface integral into a volume integral, evaluate

$$\iint_S (x^3 \, dy \, dz + y^3 \, dz \, dx + z^3 \, dx \, dy),$$

where S is the surface of $x^2 + y^2 + z^2 = 1$.

Solution: By divergence theorem,

$$\iint_S (F_1 \, dy \, dz + F_2 \, dz \, dx + F_3 \, dx \, dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx \, dy \, dz.$$

Clearly, $\vec{F}(x, y, z) = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$, then $\frac{\partial F_1}{\partial x} = 3x^2$, $\frac{\partial F_2}{\partial y} = 3y^2$, and $\frac{\partial F_3}{\partial z} = 3z^2$.

$$I = 3 \iiint_V (x^2 + y^2 + z^2) \, dx \, dy \, dz.$$

V is the closed unit sphere $x^2 + y^2 + z^2 \leq 1$. Consider the parametric form of the region by

$$x = r \cos \theta \cos \phi, \quad y = r \cos \theta \sin \phi, \quad z = r \sin \theta,$$

where $0 \leq r \leq 1$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. By the change of variable, we have

$$\begin{aligned}I &= 3 \int_{r=0}^1 \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \cdot r^2 \sin \theta \, dr \, d\theta \, d\phi \\ &= 3 \int_{r=0}^1 r^4 \, dr \int_{\theta=0}^{\pi} \sin \theta \, d\theta \int_{\phi=0}^{2\pi} d\phi = \frac{12\pi}{5}.\end{aligned}$$

Example. Evaluate

$$\iint_S (x^3 \, dy \, dz + x^2 y \, dz \, dx + x^2 z \, dx \, dy),$$

where S is the closed surface bounded by the plane $z = 0$, $z = b$, and the circle $x^2 + y^2 = a^2$.

Solution: Using Gauss divergence theorem, we have

$$\begin{aligned}I_S &= \iiint_V (3x^2 + x^2 + x^2) \, dx \, dy \, dz \\ &= 5 \int_{x=-a}^a \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^b x^2 \, dx \, dy \, dz = \frac{5}{4} \pi a^4 b.\end{aligned}$$

Example. Evaluate

$$\iint_S (x^2 dy dz + y^2 dz dx + 2z(xy - x - y) dx dy),$$

where S is the surface of the cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$.

Solution: Using Gauss divergence theorem, we have

$$\iiint_S \vec{F} \cdot \hat{n} ds = \int_{x=0}^1 \int_{y=0}^1 \int_{z=0}^1 xy dx dy dz = \frac{1}{2}.$$

Example. Evaluate $\iint_S \vec{F} \cdot \hat{n} ds$ over the entire surface of the region above xy -plane bounded by the curve $z^2 = x^2 + y^2$ and the plane $z = 4$ if $\vec{F}(x, y, z) = 4xz \hat{i} + xyz^2 \hat{j} + 3z \hat{k}$.

Solution: Using Gauss divergence theorem, we have

$$\begin{aligned} \iint_S \vec{F} \cdot \hat{n} ds &= \iiint_V (4z + xz^2 + 3) dx dy dz \\ &= \int_{z=0}^4 \int_{y=-z}^z \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} (4z + xz^2 + 3) dx dy dz \\ &= 2 \int_{z=0}^4 \int_{y=-z}^z (4z + 3) \sqrt{z^2 - y^2} dy dz \\ &= \pi \int_{z=0}^4 (4z + 3) z^2 dz = 320. \end{aligned}$$

Example. Evaluate by Green's theorem $\oint_C [(x^2 - \cosh y) dx + (y + \sin x) dy]$, where C is the rectangle with vertices $(0, 0)$, $(\pi, 0)$, $(\pi, 1)$, and $(0, 1)$.

Solution: With the comparison with Green's theorem, we have

$$\begin{aligned} M &= x^2 - \cosh y, & \frac{\partial M}{\partial y} &= -\sinh y, \\ N &= y + \sin x, & \frac{\partial N}{\partial x} &= \cos x. \end{aligned}$$

Therefore, by Green's theorem

$$\begin{aligned} \oint_C (M dx + N dy) &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \\ &= \int_{x=0}^{\pi} \int_{y=0}^1 (\cos x + \sinh y) dx dy \\ &= \pi(\cosh 1 - 1). \end{aligned}$$